Rationality constructions for cubic hypersurfaces
ICERM workshop ‘Birational Geometry and Arithmetic’

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Our focus is smooth cubic fourfolds $X \subset \mathbb{P}^5$:

1. Review recent progress on rationality
2. Place these results in the larger conjectural context
3. Propose next steps for future work

The more recent results I will present are joint with Addington, Tschinkel and Várilly-Alvarado, along with recent work of Kuan-Wen Lai.
Classical rational parametrizations
Consider a cubic fourfolds containing two disjoint planes

\[ P_1, P_2 \subset X, \quad P_i \cong \mathbb{P}^2. \]

The ‘third-point’ construction

\[ \rho : P_1 \times P_2 \dashrightarrow X \]
\[ (p_1, p_2) \quad \mapsto \quad x \]

is birational, where the line

\[ \ell(p_1, p_2) \cap X = \{p_1, p_2, x\}. \]
Writing

\[ P_1 = \{ u = v = w = 0 \} \quad P_2 = \{ x = y = z = 0 \} \]

then we have

\[ X = \{ F_{1,2}(u, v, w; x, y, z) + F_{2,1}(u, v, w; x, y, z) = 0 \}, \]

forms of bidegrees (1, 2) and (2, 1). The indeterminacy of \( \rho \) is the locus

\[ S = \{ F_{1,2} = F_{2,1} = 0 \} \subset P_1 \times P_2 \subset \mathbb{P}^8, \]

a K3 surface parametrizing lines in \( X \) meeting \( P_1 \) and \( P_2 \). These are blown down by \( \rho^{-1} \).
Cubic fourfolds containing quartic scrolls

This example is due to Morin-Fano (1940) and Beauville-Donagi (1985).
A quartic scroll is a smooth surface

\[ T_4 \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^5 \]

embedded via forms of bidegree \((1, 2)\). The linear system of quadrics cutting out \( T_4 \) collapses all its secant lines, inducing a map

\[ \mathbb{P}^5 \dashrightarrow Q \subset \mathbb{P}^5 \]

onto a hypersurface of degree two. Any cubic fourfold

\[ X \supset T_4 \]

is mapped birationally to \( Q \) and thus is rational.
What is the parametrizing map \( \rho : Q \to X? \)

Fix a point on a degree 14 K3 surface

\[ s \in S \subset \mathbb{P}^8 \]

and take a double (tangential) projection of \( \text{Bl}_s(S) \subset \mathbb{P}^5 \). The resulting surface is contained in a quadric hypersurface \( Q \) and \( \rho \) arises from the cubics containing this surface. Again, we have a K3 surface.
A cubic fourfold with double point

\[ x_0 = [1, 0, 0, 0, 0, 0] \in X \subset \mathbb{P}^5 \]

is always rational via projection from \( x_0 \)

\[ X \sim \mathbb{P}^4. \]

The inverse map \( \rho \) blows up a K3 surface

\[ S = \{ F_2(v, w, x, y, z) = F_3(v, w, x, y, z) = 0 \} \]

where \( X = \{ uF_2 + F_3 = 0 \} \).
Classification and conjectures
Moduli space

Let \( C \) denote the moduli space of cubic fourfolds, smooth (as a stack) of dimension 20. The middle Hodge numbers are

\[
\begin{array}{cccccc}
0 & 1 & 21 & 1 & 0
\end{array}
\]

Voisin has shown that the period map for cubic fourfolds is an open immersion into its period domain, a type IV Hermitian symmetric domain – analogous to K3 surfaces. When \( X \) is a very general cubic fourfold we have

\[
H^{2,2}(X) \cap H^4(X, \mathbb{Z}) = \mathbb{Z} h^2
\]

where \( h \) is the hyperplane class. Cubic fourfolds with

\[
H^{2,2}(X) \cap H^4(X, \mathbb{Z}) \supset \mathbb{Z} h^2
\]

are special.
Speciality Conjecture

Conjecture (Harris-Mazur ??)

*All rational cubic fourfolds are special.*

The special cubic fourfolds form a countably infinite union of irreducible divisors

\[ \bigcup_d C_d \subset C \]

where \( d \equiv 0, 2 \pmod{6} \) and \( d \geq 8 \), e.g.,

- \( d = 8: X \supset P \) a plane;
- \( d = 14: X \supset T_4 \) a quartic scroll.
While no cubic fourfolds are *known* to be irrational most people doubt that *all* special cubic fourfolds are rational. I would personally be very surprised if the examples

- $d = 12$: $X \supset T_3 \simeq F_1$ a cubic scroll;
- $d = 20$: $X \supset V \simeq \mathbb{P}^2$ a Veronese surface;

were generally rational. Hence we narrow the search.

All known rational parametrization $\rho : \mathbb{P}^4 \dashrightarrow X$ blow up a K3 surface.
Cubic fourfolds and K3 surfaces

On blowing up a smooth surface $S$ in a fourfold $Y$, we have

$$H^4(\text{Bl}_S(Y), \mathbb{Z}) = H^4(Y, \mathbb{Z}) \oplus H^2(S, \mathbb{Z})(-1)$$

where the $(-1)$ reflects Tate twist. This motivates the following:

**Definition**

A polarized K3 surface $(S, f)$ is associated with a cubic fourfold $X$ if we have a saturated embedding of the primitive Hodge structure

$$H^2(S, \mathbb{Z}) \circ (-1) \hookrightarrow H^4(X, \mathbb{Z}).$$

It follows that $X$ is special.
Some basic properties:

- A general cubic fourfold $[X] \in C_d$ admits an associated K3 surface unless $4|d$, $9|d$, or $p|d$ for some odd prime $p \equiv 2 \pmod{3}$;
- All known rational cubic fourfolds admit associated K3 surfaces;
- Kuznetsov proposed an alternate formulations via derived categories of coherent sheaves – Addington and Thomas have shown this is equivalent to the Hodge characterization over dense open subsets of each $C_d$;
- Distinct polarized K3 surfaces $(S_1, f_1)$ and $(S_2, f_2)$ may have isomorphic primitive cohomologies – this characterizes derived equivalence among rank one K3 surfaces.
Thus associated K3 surfaces are far from unique; the monodromy representation over $C_d$ when $3|d$ precludes a well-defined choice!
Is there a diagram

\[
\begin{array}{c}
\text{$X$} \\
\mathbb{P}^4 & \mathbb{P}^4 \\
\beta_1 & \beta_2
\end{array}
\]

where $X$ is a cubic fourfold, $\beta_i$ blows up a K3 surface $S_i$, but $S_1$ and $S_2$ are distinct? We would expect the K3 surfaces to be derived equivalent if the only other cohomology is of Hodge-Tate type.

Lai and I have found such diagrams for more general Fano fourfolds.
A stronger conjecture

Conjecture (Kuznetsov* Conjecture)

A cubic fourfold is rational if and only if it admits an associated K3 surface.

Kuznetsov originally expressed this in derived category language. Addington-Thomas – taken off-the-shelf – applies to dense open subsets of the appropriate $C_d$. The recent theorem by Kontsevich and Tschinkel on specialization of rationality implies the statement above.

Question

Is the derived category condition in Kuznetsov’s conjecture stable under smooth specialization?

A proof was recently announced by Arend Bayer.
Cubic fourfolds and twisted K3 surfaces

Definition

A polarized K3 surface \((S, f)\) is twisted associated with a cubic fourfold \(X\) if we have inclusions of Hodge structures

\[
H^2(S, \mathbb{Z}) \circ (-1) \leftrightarrow \Lambda \xrightarrow{j} H^4(X, \mathbb{Z})
\]

where \(j\) is saturated and \(\iota\) has cyclic cokernel. 

\(\Lambda\) is characterized as the kernel of a homomorphism

\[
\alpha : H^2(S, \mathbb{Z}) \circ \rightarrow \mathbb{Q}/\mathbb{Z},
\]

the twisting data when \(\text{Pic}(S) = \mathbb{Z}f\). Huybrechts has shown a general \([X] \in \mathcal{C}_d\) admits a twisted associated K3 if and only if

\[
d/2 = \prod_{i} p_i^{n_i}
\]

where \(n_i\) is even when \(p_i \equiv 2 \pmod{3}\).
Examples motivated by the classification
Tabulation of discriminants

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Twisted structures and rationality

The first result goes back to the 1990’s:

**Theorem**

*Each* $X \in \mathcal{C}_8$, *containing a plane* $P$, *yields a twisted K3 surface* $(S, f, \alpha)$ *of degree two and order two. $X$ is rational when $\alpha$ vanishes in $\text{Br}(S)$.*

*Idea:* projecting from $P$ gives a quadric surface bundle $\text{Bl}_P(X) \to \mathbb{P}^2$ which is rational when the Brauer class vanishes.

The second is more recent

**Theorem (AHTV 2016)**

$X \in \mathcal{C}_{18}$ *yields a twisted K3 surface* $(S, f, \alpha)$ *of degree two and order three. $X$ is rational when $\alpha$ vanishes in $\text{Br}(S)$.*

*Idea:* Fiber in sextic del Pezzo surfaces.
**Challenge:** Give more examples along these lines, especially for higher torsion orders.
The case of \( d = 50 \) looks quite intriguing. How can we make sense of five torsion?
The fibrations in surfaces we use do not obviously generalize:

Does there exist a class of geometrically rational surfaces \( \Sigma/K \) (say, \( K = \mathbb{C}(\mathbb{P}^2) \)) whose rationality over \( K \) is controlled by an element \( \alpha \in \text{Br}(L) \) with order prime to 6, where \( L/K \) is a finite extension depending on \( \Sigma \)?
Here are new and surprising results:

**Theorem (Russo-Staglianò 2017)**

$X \in C_{26}$, containing a septic scroll with three transverse double points, is rational.

$X \in C_{38}$, containing a degree-ten surface isomorphic to $\mathbb{P}^2$ blown up in ten points, is rational.

These are the first new divisorial examples predicted by Kuznetsov, which looks much more plausible than a year ago.

The construction uses families of conics 5-secant to a prescribed surface; the family $B$ happens to be rational. Each of these meets a cubic fourfold in six points, so the residual point of intersection gives $B \sim X$. 
Parametrization questions

**Challenge:** Describe the parametrization $\rho : \mathbb{P}^4 \rightarrow X$ in the Russo-Staglianò examples. Does it blow up an associated K3 surface? Give explicit linear series on $X$ inducing $\rho^{-1}$.

**Question**

*Can the rationality construction be extended to $d = 42$? (Lai)*

*Are there rationality constructions associated with degree $e$ rational curves $(3e - 1)$-secant to a suitable surface? (Yes for $e = 1, 2$!)*